

1. Light-atom interaction Hamiltonian

1.1 Quasi-classical Hamiltonian, Light and 2 level atom

Here is derivation in terms of hand-written notes

1.2 Applying rotating wave approximation

We can continue from the Hamiltonian of light-atom interaction, where light is described by Rabi frequency and has central frequency ω and phase ϕ

$$\hat{H} = \frac{\hbar\omega_a}{2}\sigma_z + \hbar\Omega \cdot \sigma_x \cos(\omega t + \phi)$$

By going into the rotating frame,

$$\hat{H}_{\text{rot. frame}} = \hat{U}\hat{H}\hat{U}^\dagger + i\hat{U}^\dagger \frac{d\hat{U}}{dt}, \text{ where } \hat{U} = e^{i\omega t \hat{\sigma}_z/2} = \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}$$

And using the Baker-Campbell-Hausdorf formula

$$e^{i\lambda\hat{G}}\hat{A}e^{-i\lambda\hat{G}} = \hat{A} + i\lambda[\hat{G}, \hat{A}] + \frac{(i\lambda)^2}{2!}[\hat{G}[\hat{G}, \hat{A}]] + \dots$$

, we find the first term $\hat{U}\hat{H}\hat{U}^\dagger$ with $\lambda = \frac{\omega t}{2}$:

$$\begin{aligned} e^{i\lambda\sigma_z}\sigma_x e^{-i\lambda\sigma_z} &= \sigma_x + i\lambda(2i)\sigma_y + \frac{(i\lambda)^2}{2!}(2i)(-2i)\sigma_x + \frac{(i\lambda)^3}{3!}(2i)(-2i)(2i)\sigma_y + \frac{(i\lambda)^4}{4!}(2i)(-2i)(2i)(-2i)\sigma_x + \dots \\ &= \sigma_x + i\lambda(2i)\sigma_y + \frac{(i\lambda)^2}{2!}(2)^2\sigma_x + \frac{(2\lambda)^3}{3!}i^4\sigma_y + \frac{(i\lambda)^4}{4!}(2)^4\sigma_x + \dots \end{aligned}$$

Collecting series together, using known expansions $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $i^{2n} = (-1)^n$:

$$e^{i\lambda\sigma_z}\sigma_x e^{-i\lambda\sigma_z} = (i2\lambda)^{2n} \frac{\sigma_x}{(2n)!} + i^{2n+2}(2\lambda)^{2n+1} \frac{\sigma_y}{(2n+1)!} = \cos(\omega t)\sigma_x - \sin(\omega t)\sigma_y = \begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}$$

The Hamiltonian in the rotating frame:

$$\begin{aligned} \hat{H}_{\text{rot. frame}} &= \frac{\omega_a}{2}\sigma_z + \Omega \left(\hat{U}\hat{\sigma}_x\hat{U}^\dagger \right) \cos(\omega t + \phi) + i\frac{\omega t}{2}\sigma_z = \frac{\delta}{2}\sigma_z + \Omega (\cos(\omega t)\hat{\sigma}_x - \sin(\omega t)\hat{\sigma}_y) \cos(\omega t + \phi) = \\ &= \frac{\delta}{2}\sigma_z + \frac{\Omega}{2}(\cos(2\omega t + \phi) + \cos(\phi))\hat{\sigma}_x - \frac{\Omega}{2}(\sin(2\omega t + \phi) - \sin(\phi))\hat{\sigma}_y = \end{aligned}$$

We neglect terms which oscillate twice electric field frequency and arrive to Hamiltonian in rotating frame and with rotating-wave approximation applied

$$\hat{H}_{\text{RWA}} = \frac{\delta}{2}\sigma_z + \frac{\Omega}{2}(\cos(\phi)\sigma_x + \sin(\phi)\sigma_y)$$

where $\delta = \omega_a - \omega$.

We can rewrite it in the form of Bloch vector

$$\hat{H}_{\text{RWA}} = \frac{\hbar}{2} (\vec{m} \cdot \vec{\sigma}) \quad \text{Eq. *}$$

2. Density matrix and Bloch sphere

2.1 Pure state and arbitrary superposition

If we start with a generic supersposition of two states which describe a two-level atom
We describe our atom using two basis states $|0\rangle, |1\rangle$. Any pure state can be written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$$

Because our atom is not lost from the system, we have normalization condition

$$|\alpha|^2 + |\beta|^2 = 1,$$

and because global phase is irrelevant, the state can be parameterized as

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi} \sin\left(\frac{\theta}{2}\right)|1\rangle.$$

So every pure state is specified by two parameters or two angles.

2.2 Introducing a density matrix

Density matrix will be spanned as such

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$$

where diagonal elements are populations of unperturbed states $|0\rangle, |1\rangle$. Off diagonal terms contain coherence.

In density-matrix language,

$$\rho = \frac{1}{2} (I + r_x \sigma_x + r_y \sigma_y + r_z \sigma_z).$$

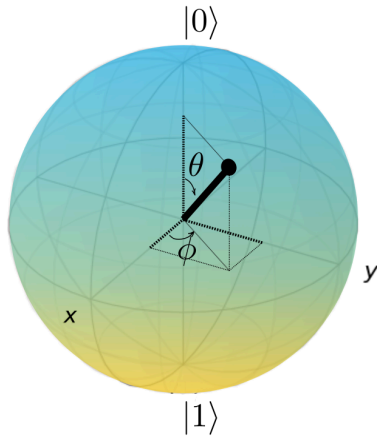
Let's look where this Bloch vector \mathbf{r} points if we have some pure states?

1. Let's prepare pure state $|0\rangle$. Detailed answer for this situation only

$$\rho = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} + \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

so $\mathbf{r} = (0, 0, 1)$. Analogously, for state $|1\rangle$ it get's us to $\mathbf{r} = (0, 0, -1)$. We can start placing our findings onto

a three-dimensional axes.

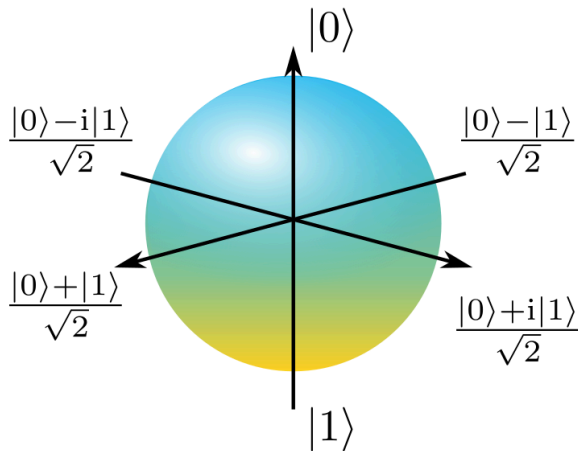


2. Let's see which states are located on equator $|+x\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$. And corresponding density matrix:

$$\rho = | + x \rangle \langle + x | = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} + \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

Indeed Bloch vector points along x axis $\mathbf{r} = (1, 0, 0)$.

In the end we can span all available axis with superpositions of basis states $| - x \rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$ and $|\pm y\rangle = \frac{|0\rangle \pm i|1\rangle}{\sqrt{2}}$



Okay if we come back once again to our generic form of eigenstate of the two-level atoms

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi} \sin\left(\frac{\theta}{2}\right)|1\rangle.$$

We write down density matrix

$$\rho = |\psi\rangle \langle \psi| = \begin{pmatrix} \cos^2\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) e^{i\phi} \\ \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) e^{-i\phi} & \sin^2\left(\frac{\theta}{2}\right) \end{pmatrix}$$

After a few simple math exercises and using Euler equation $e^{i\phi} = \cos \phi + i \sin \phi$

$$\rho = \begin{pmatrix} (1 + \cos \theta)/2 & \sin \theta e^{i\phi}/2 \\ \sin \theta e^{-i\phi}/2 & (1 - \cos \theta)/2 \end{pmatrix} = 1/2(I + \sin \theta(\cos(\phi)\sigma_x + \sin(\phi)\sigma_y) + \cos \theta \sigma_z)$$

All the pure states live on the surface of the sphere, pointed by a Bloch vector by two coordinates or by a vector in spherical coordinates $\mathbf{r} = (\sin \theta \cos(\phi), \sin \theta \sin(\phi), \cos \theta)$

$$\rho = 1/2(I + \mathbf{r} \cdot \sigma) \quad \text{Eq. **}$$

There are three important conclusions which we need to make.

We can now derive a bit more general rules (1) pure states live on the surface of the sphere (2) Bloch vector can not be longer than the unity (3) mixed states have take inner point in the ball.

3. Evolution under applied field

Let's start with the Liouville equation

$$i\hbar \frac{d\rho}{dt} = [H, \rho]$$

and we take both Hamiltonian and density matrices to be superpositions of Pauli matrices Eqs (* and **) /

Let's derive one collorary, which we will need further, let's see how we can rewrite product of Pauli matrices and two vectors:

$$(\mathbf{a} \cdot \sigma)(\mathbf{b} \cdot \sigma) = \sum_{ij} a_i b_j \sigma_i \sigma_j = \sum_{ij} a_i b_j (\delta_{ij} I + i\epsilon_{ijk} \sigma_k) = (\mathbf{a} \cdot \mathbf{b})I + i(\mathbf{a} \times \mathbf{b}) \cdot \sigma$$

Let's use this equation in the Liouville equation.

RHS:

$$[H, \rho] = \hbar/2[\mathbf{m}\sigma, 1/2(I + \mathbf{r} \cdot \sigma)] = \hbar/4[\mathbf{m}\sigma, (\mathbf{r} \cdot \sigma)] = \hbar/4((\mathbf{m} \cdot \mathbf{r})I + i(\mathbf{m} \times \mathbf{r}) \cdot \sigma - (\mathbf{r} \cdot \mathbf{m})I - i(\mathbf{r} \times \mathbf{m}) \cdot \sigma)$$

$$[H, \rho] = \hbar/2(\mathbf{m} \times \mathbf{r}) \cdot \sigma$$

Combining both sides together

$$i\hbar \frac{d\mathbf{r}}{dt} \sigma = i\hbar/2(\mathbf{m} \times \mathbf{r}) \cdot \sigma$$

Finally we arrive to final equation of how density matrix pointing vector changes under Hamiltonian:

$$\frac{d\mathbf{r}}{dt} = 1/2(\mathbf{m} \times \mathbf{r})$$

What does this equation describe? We can rewrite vector r as having collinear and orthogonal components with m. From the form of the equation we see that colinear part would zero RHS and thus stay unchanged under Hamiltonian. Orthogonal components would initiate circular motion/ precession around vector m. This could be visible from the diagram. Larmor precession.

4. Take aways

- Hamiltonian was derived in dipole approximation for quantized atom with two levels
- We introduced arbitrary states for two-level atom
- We derived equation of motion of density matrix under the light-atom interaction Hamiltonian
- We notices that state would perform precession around vector of Hamiltonian
- Next lecture: Two-level atom as a qubit and tomography.